Probability

3. Combinatorics

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Foreword

- To compute probabilities, we often need to take a detour through **combinatorics**, whose goal is to **count the possible outcomes** of a random experiment.
- Many probability problems are, at their core, counting problems:
  - a lottery player may want to know how many different tickets are possible,
  - a statistician may want to determine how many distinct samples can be formed.
- Counting is central to the **classical (naive) definition of probability**:

$$\mathbb{P}(A) = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}},$$

  which applies when all outcomes are equally likely.
- In many cases, this counting can be carried out using simple tools such as **permutations** and **combinations**.
- This chapter is mainly based on Chapter 3.2 of Newbold et al. (2023), Chapter 2 of Elsner Bernhard (2021), and Chapters 1.3–1.4 of Blitzstein and Hwang (2019).

1. Naive Definition of Probability

Naive (Classical) Definition of Probability

Definition 1. Naive definition of probability

Let A be an event associated with a random experiment with a finite sample space Ω. The naive probability of A is defined as

$$\mathbb{P}(A) = \frac{\textit{card}(A)}{\textit{card}(\Omega)} = \frac{\textit{number of outcomes favorable to } A}{\textit{total number of possible outcomes in } \Omega}.$$

- This definition is simple and intuitive: probability is a **proportion**.
- However, it is **restrictive**:
  - the sample space Ω must be **finite**;
  - each outcome must carry the **same probability mass**.

When Does the Naive Definition Apply?

The naive definition of probability can be used in the following situations:

- **Symmetry**:
  - the structure of the experiment makes all outcomes interchangeable;
  - examples: fair dice, fair coins, random shuffling.
- **Equal likelihood by design**:
  - outcomes are made equally likely by a randomization mechanism;
  - examples: random draws, lotteries, randomized assignments.
- **Benchmark or null model**:
  - the naive probability is used as a reference point;
  - it helps detect deviations from randomness or fairness.
  - example: assuming a fair die with $\mathbb{P}(1) = \cdots = \mathbb{P}(6) = \frac{1}{6}$, and comparing observed frequencies to this benchmark.

Example: Computing a Probability with the Naive Approach

Rolling a Die

Consider the roll of a fair six-sided die.

- The sample space is
$$\Omega = \{1, 2, 3, 4, 5, 6\}.$$
- Let A be the event "*the outcome is even*":
$$A = \{2, 4, 6\}.$$
- Using the naive definition,
$$\mathbb{P}(A) = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}} = \frac{3}{6} = \frac{1}{2}.$$
- Your turn: what is the probability of the event B: "*the outcome is either 1 or 3*"?

2. Counting

An Example

Example

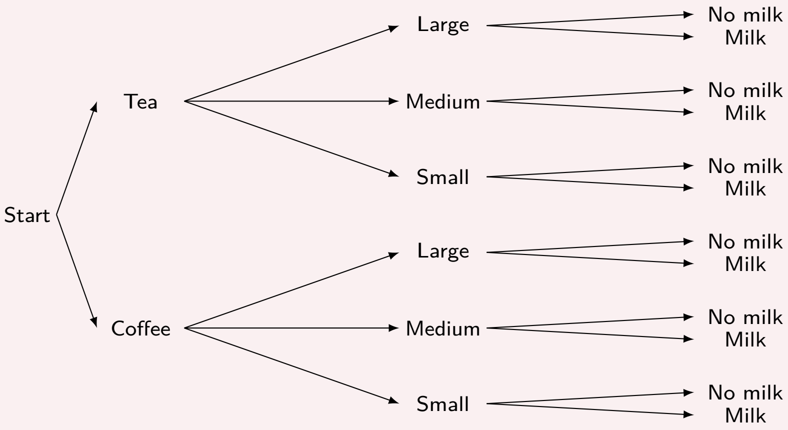
Consider a person who makes a sequence of three choices in a coffee shop:

- *first, chooses a drink: {coffee, tea} (2 choices);*
- *then, chooses a size: {small, medium, large} (3 choices);*
- *finally, chooses milk: {yes, no} (2 choices).*

We assume that each choice is made independently of the others.

The Example, With a Tree Diagram

1st choice ($n_1 = 2$) **2nd choice ($n_2 = 3$)** **3rd choice ($n_3 = 2$)**



We observe that the total number of possible outcomes is $2 \times 3 \times 2 = 12$.

Multiplication Principle

The example from the previous slide illustrates the **multiplication principle**.

Definition 2. Multiplication Principle

- *If a task consists of p steps,*
- *and the k-th step can be done in n_k ways,*
- *then the task can be performed in $n_1 \times n_2 \times \cdots \times n_p$ ways.*
- *The idea is to use **multiplication** when choices are made **step by step**.*

Same Example, Different Order of Choices

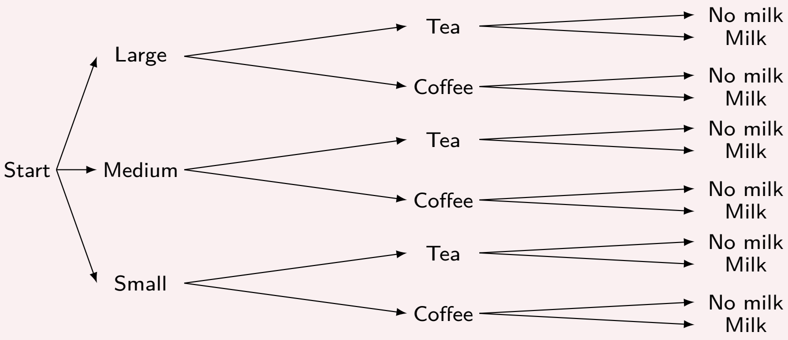
Same Example, Different Order of Choices

*Consider the **same experiment**, but with a different order in which the choices are made:*

- *first choose the **size**,*
- *then the **drink**,*
- *finally whether to add **milk**.*

Same Example, Different Order of Choices

1st choice (3 sizes) **2nd choice (2 drinks)** **3rd choice (2 options)**



The total number of possible outcomes is $3 \times 2 \times 2 = 12$.

The Total Number Does Not Change

- The experiment itself has not changed:
  - a drink,
  - a size,
  - milk or no milk.
- Only the **order in which we describe the choices** has changed.
- Each complete path in the tree still corresponds to **one unique outcome**.
- The number of outcomes depends on **how many choices are available at each step**, not on the order in which the steps are performed.

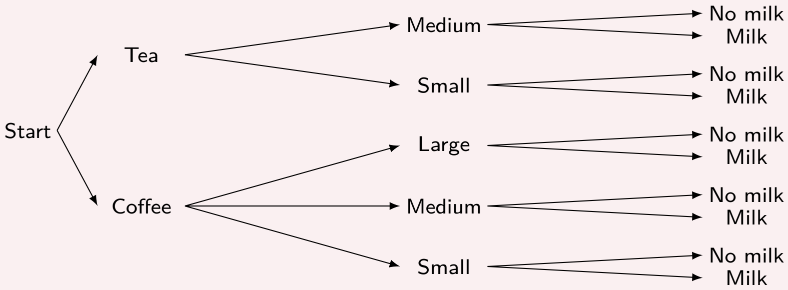
Another Example

Another Example

Let us now consider a restriction with respect to the first example: the large format for tea is not offered by the coffee shop.
This creates a tree diagram which is no longer symmetric.

The Other Example, With a Tree Diagram

1st choice ($n_1 = 2$) **2nd choice (n_2 depends)** **3rd choice ($n_3 = 2$)**



The tree is no longer symmetric: tea does not come in a large format.

$$\text{Total outcomes} = \underbrace{(1 \times 2 \times 2)}_{\text{if tea}} + \underbrace{(1 \times 3 \times 2)}_{\text{if coffee}} = 4 + 6 = 10.$$

Addition Principle

The previous example illustrates the **addition principle**.

Definition 3. Addition Principle

- *If a task can be performed in p **mutually exclusive** different ways,*
- *and the k-th way can be performed in n_k ways,*
- *then the task can be performed in $n_1 + n_2 + \cdots + n_p$ ways.*
- *The idea is to use **addition** when choices are made **either/or**.*

2.1. Ordering

Orderings: People in a Line

Example

Suppose n people are waiting to form a line.

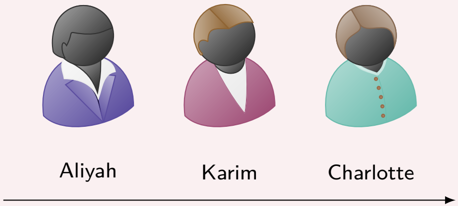
- *Each person appears exactly once in the line.*
- *How many different **orderings** (line-ups) are possible?*

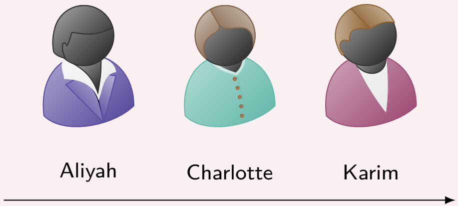
For now, consider that there are only three people: Aliyah, Karim, and Charlotte.

Aliyah Karim Charlotte

Ordering: the Order is Important

- In this example, the elements are used **once**,
- And the **order** is important:
  - the line formed by Aliyah, Karim, Charlotte,
  - is different from the line formed by Aliyah, Charlotte, Karim.





Orderings: Step-by-Step Reasoning

To form a line of three distinct people:

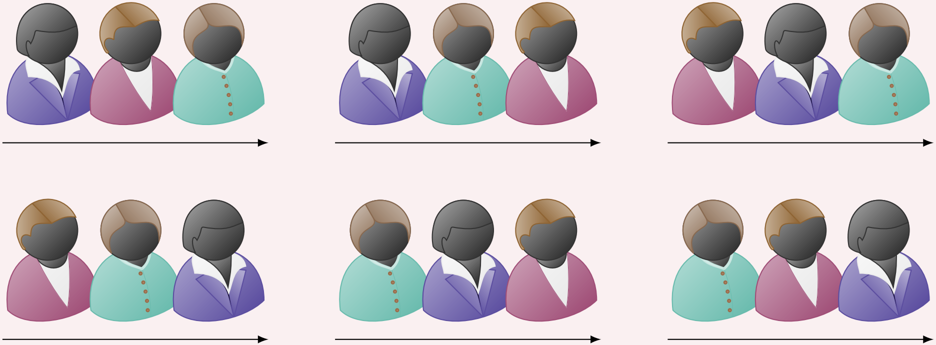
- For the **first position**, we can choose **any one of the three** individuals.
- For the **second position**, there are **two remaining** individuals to choose from.
- For the **third position**, there is **only one** individual left.

By the multiplication principle,

$$3 \times 2 \times 1 = 6$$

different orderings are possible.

Three People in a Line: All Possible Orderings



For three distinct people, there are 6 different possible orderings.

Orderings of n Distinct Objects

Now, suppose we want to arrange n distinct objects in a sequence.

- For the **first position**, there are n possible choices.
- For the **second position**, there are $n-1$ remaining choices.
- $\vdots$
- For the **last position**, there is only 1 choice left.

By the multiplication principle, the number of possible orderings is

$$n \times (n-1) \times \cdots \times 1 = n!.$$

This quantity is called the **factorial** of n.

Note: recall that $0! = 1$: there is only one way to order nothing.

Exercise

Exercise

Simplify the following expressions:

1. $\dfrac{10!}{8!}$
2. $\dfrac{10!}{12!}$
3. $\dfrac{n!}{(n-1)!}$
4. $\dfrac{n!}{(n-3)!}$

Solution

1.
$$\frac{10!}{8!} = \frac{10 \times 9 \times 8!}{8!} = 10 \times 9 = 90.$$

2.
$$\frac{10!}{12!} = \frac{10!}{12 \times 11 \times 10!} = \frac{1}{12 \times 11} = \frac{1}{132}.$$

3.
$$\frac{n!}{(n-1)!} = \frac{n \times (n-1)!}{(n-1)!} = n.$$

4.
$$\frac{n!}{(n-3)!} = \frac{n \times (n-1) \times (n-2) \times (n-3)!}{(n-3)!} = n(n-1)(n-2).$$

Tip

When simplifying factorials, expand the larger factorial until the smaller one appears and cancels.

2.2. Permutations Without Repetitions

Permutations Without Repetition

Orderings are the building blocks of permutations.

Definition 4. Permutation

*A **permutation** of n distinct objects is an **ordering** of all n objects, where each object appears **exactly once** and **order matters**.*

- *Permutations describe situations where **order matters**.*
- *The number of permutations of n distinct objects is*

$$n \times (n-1) \times \cdots \times 1 = n!.$$

2.3. Permutations With Repetition

Permutations With Repetition: An Example

Example

An iPhone passcode consists of 6 digits.

- *Each digit can be any number from $\{0, 1, \ldots, 9\}$.*
- *The **order matters**.*
- *The same digit may appear **several times**.*

How many different passcodes codes are possible?

Counting Passcodes

- There are 10 possible choices for each digit.
- The choice for each position is made independently.
- By the multiplication principle,

$$10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^6$$

different passcodes are possible.

Note

- *Each passcode is an ordered sequence of length 6.*
- *Repetition comes from* ***reusing digits****.*

Permutations With Repetition (Sequences)

Definition 5. Permutation with repetition

A ***permutation with repetition*** *is an ordered sequence of length p formed from n distinct objects, where the same object may be chosen more than once.*

- *Each position can be filled independently.*
- *The number of such permutations is*

$$n^p$$

Exercise: Sampling With Replacement

Exercise

An urn contains six balls numbered from 1 to 6. We draw one ball ***four times with replacement****.*

A ***draw*** *is defined as the* ***ordered sequence*** *of the four results. For example, the draw 4–2–3–3 means that we first draw 4, then 2, then twice 3.*

How many different draws are possible?

Solution: Sampling With Replacement

- Each draw has 6 possible outcomes.
- Because the draws are made **with replacement**, the same number may appear more than once.
- The order of the results **matters**.
- The four draws are independent.

This is a **permutation with repetition**. The number of different draws is:

$$6^4 = 1296.$$

2.4. Permutations of a Multiset

Permutations of a Multiset: An Example

Example

Consider the word BANANA.

- *It contains 6 letters in total.*
- *Some letters are **repeated**:*
  - *A appears 3 times,*
  - *N appears 2 times,*
  - *B appears 1 time.*
- *How many **distinct orderings** of the letters can we form?*

$n!$ Overcounts

- If all letters were distinct, there would be $6!$ orderings.
- However, swapping identical letters does not create a new ordering.
- For example:
  - exchanging the three A's does not create a new word;
  - exchanging the two N's does not create a new word.
- Therefore, $6!$ counts the same ordering multiple times.

Correct Counting

- Start from the total number of orderings: $6! = 720$.
- Treating all letters as distinct, each arrangement is counted once.
- However:
  - for any fixed arrangement, the three A's can be permuted among themselves in $3!$ ways,
  - these $3!$ permutations all produce the **same word**,
  - similarly, the two N's can be permuted in $2!$ ways without changing the word.
- Therefore, each distinct word is counted $3! \times 2!$ times in $6!$.
- Since each distinct word is counted several times, we divide the number of all possible orderings (when treating identical letters as different) by how many times we overcounted.
- The correct number of distinct orderings is

$$\frac{6!}{3!\,2!}.$$

Permutations of a Multiset

Definition 6. Permutation of a Multiset

*A **permutation of a multiset** is an ordering of n objects, where some objects may be **identical**.*

- *Suppose there are n objects in total.*
- *Among them, there are $n_1, n_2, \ldots, n_k$ identical objects of each type.*
- *The number of distinct permutations is*

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}.$$

Exercise

Exercise

A shelf contains 9 mugs:

- 4 *red mugs,*
- 3 *blue mugs,*
- 2 *green mugs.*

Mugs of the same colour are ***indistinguishable****.*

In how many different ways can the 9 mugs be arranged on the shelf?

Solution

- There are 9 positions on the shelf to be filled.
- If all mugs were distinct, there would be $9!$ possible orderings.
- However:
  - exchanging the 4 red mugs does not change the arrangement,
  - exchanging the 3 blue mugs does not change the arrangement,
  - exchanging the 2 green mugs does not change the arrangement.
- Each distinct arrangement is therefore counted $4! \times 3! \times 2!$ times in $9!$.

This corresponds to a **permutation of a multiset**. The number of distinct orderings is:

$$\begin{aligned}
\frac{9!}{4!\,3!\,2!} &= \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4!}{4!\,3! \times 2} \\
&= \frac{9 \times 2 \times 4 \times 7 \times 2 \times 3 \times 5}{3 \times 2 \times 2} \\
&= 9 \times 4 \times 7 \times 5 = 1260.
\end{aligned}$$

2.5. Partial Permutations (Arrangements)

Partial Permutations: An Example

Example

From a group of n people, we want to pick p people to fill p distinct positions (e.g., President, Secretary, Treasurer, ...).

- *Each person can be used at most once.*
- *The* ***order matters*** *(the role matters).*

How many outcomes are possible?

Counting Partial Permutations

- We fill positions **one by one**:
  - n choices for the first position,
  - $n-1$ choices for the second,
  - $\vdots$
  - $n-p+1$ choices for the p-th.
- By the multiplication principle, the number of ordered selections is

$$n \times (n-1) \times \cdots \times (n-p+1) = \frac{n!}{(n-p)!}.$$

Partial Permutations (Arrangements)

Definition 7. Arrangement

*An **arrangement** of p objects chosen from n distinct objects is an **ordered selection of length** p where no object is used more than once.*

- *The number of arrangements is*

$$A_n^p = \frac{n!}{(n-p)!}.$$

Exercise: Sampling Without Replacement

Exercise

An urn contains 50 *balls numbered from* 1 *to* 50.

We draw 5 *balls* ***without replacement****, which produces a sequence. For example:* $41–50–47–15–34$.

The order in which the balls appear in the sequence is relevant.

How many different outcomes are possible?

Solution

- There are $n = 50$ distinct balls.
- We draw $p = 5$ balls **without replacement**.
- Each ball can appear **at most once**.
- The **order matters**, since the outcome is an ordered sequence.

This is an arrangement of 5 objects chosen from 50. The number of possible outcomes is:

$$
\begin{aligned}
A_{50}^{5} &= \frac{50!}{(50-5)!} \\
&= \frac{50!}{45!} \\
&= \frac{50 \times 49 \times 48 \times 47 \times 46 \times 45!}{45!} \\
&= 254,251,200.
\end{aligned}
$$

2.6. Combinations without repetition

Combinations Without Repetition: An Example

Example

Suppose we want to select 3 people from a group of 5:

{Aliyah, Charlotte, Jean-Damien, Karim, Lily-Rose}.

- *Each person can be selected **at most once**.*
- *The **order does not matter**:*
  - *selecting (Aliyah, Karim, Charlotte)*
  - *is the same as (Charlotte, Karim, Aliyah).*
- *How many different groups of 3 people can be formed?*

From Orderings to Combinations

- First, let us count all **orderings** of 3 people chosen from 5.
- Step-by-step:
  - 5 choices for a first person,
  - 4 choices for a second person,
  - 3 choices for a third person.
- This gives

$$5 \times 4 \times 3 = \frac{5!}{(5-3)!}$$

ordered selections.

Dividing by $3!$

- In the previous count, we assumed that **order matters**.
- But, for combinations, **order does not matter**.
- For any fixed group of 3 people:
  - there are $3!$ different orderings of the same group;
  - all these orderings correspond to the **same combination**.
- Therefore, each group is counted $3!$ times.
- To count each group once, we divide by $3!$.
- The number of distinct groups of 3 people chosen from 5 is

$$\frac{5 \times 4 \times 3}{3!} = \frac{5!}{3!(5-3)!}.$$

Combinations Without Repetition

Definition 8. Combination

*A **combination** of p objects chosen from n distinct objects is a selection where:*

- *each object is used at most once,*
- *the **order does not matter**.*

The number of such combinations is

$$\binom{n}{p} = \frac{n!}{p!(n-p)!}.$$

*The quantity $\binom{n}{p}$ is called a **binomial coefficient**.*

Notation

The notation can either be $\binom{n}{k} = C_k^n = {}_nC_k$

Exercise

Exercise (from McClave et al., 2022)

Suppose you plan to invest equal amounts of money in each of five business ventures.

If you have 20 ventures from which to make the selection, how many different samples of five ventures can be selected from the 20?

Solution

- There are $n = 20$ distinct ventures, and we want to select $p = 5$ of them.
- Each venture can be selected **at most once**.
- Since the investment is split equally, the **order does not matter**: choosing $\{v_2, v_4, v_6, v_8, v_{10}\}$ is the same portfolio as $\{v_{10}, v_8, v_6, v_4, v_2\}$.
- Therefore, this is a **combination without repetition**, and the number of possible portfolios is

$$\binom{20}{5} = \frac{20!}{5!\,15!} = \frac{20 \times 19 \times 18 \times 17 \times 16}{5 \times 4 \times 3 \times 2 \times 1} = 15{,}504.$$

2.7. A Few Properties of Binomial Coefficients

Properties of Binomial Coefficients

Properties of Binomial Coefficients

For all integers $n \geq p$:

- ***Symmetry***:

$$\binom{n}{p} = \binom{n}{n-p}.$$

- ***Boundary cases***:

$$\binom{n}{0} = \binom{n}{n} = 1.$$

- ***Pascal's formula***:

$$\binom{n}{p} = \binom{n-1}{p-1} + \binom{n-1}{p}.$$

Proof: Symmetry of Binomial Coefficients

- Starting from the definition:

$$\binom{n}{p} = \frac{n!}{p!(n-p)!}.$$

- Swap p and $n-p$:

$$\binom{n}{n-p} = \frac{n!}{(n-p)!(n-(n-p))!} = \frac{n!}{(n-p)!\,p!}.$$

- Since multiplication is commutative $(p!(n-p)! = (n-p)!\,p!)$,

$$\binom{n}{n-p} = \binom{n}{p}.$$

Symmetry of Binomial Coefficients: Intuition

Consider choosing a committee of size p from a group of n people.

- There are $\binom{n}{p}$ ways to choose the committee.
- Alternatively, we can choose the $n - p$ **people who are not on the committee**.
- Specifying who is on the committee uniquely determines who is not, and vice versa.
- These are two different ways of counting the **same set of outcomes**.

Therefore,

$$\binom{n}{p} = \binom{n}{n-p}.$$

Proof: Boundary Cases

- Recall:

$$\binom{n}{p} = \frac{n!}{p!(n-p)!}.$$

- For $p = 0$ (choose non of the n objects):

$$\binom{n}{0} = \frac{n!}{0!\,n!} = \frac{n!}{1 \cdot n!} = 1.$$

- For $p = n$ (choose all of the n objects):

$$\binom{n}{n} = \frac{n!}{n!\,0!} = \frac{n!}{n! \cdot 1} = 1.$$

Pascal's Identity: Intuition (not a formal proof)

Pascal's Identity write:

$$\binom{n}{p} = \binom{n-1}{p-1} + \binom{n-1}{p}.$$

Suppose we want to choose p people from a group of n people, with $p \leq n$.

Fix one particular person, say **Aliyah**.

- Any group of p people:
  - either **includes Aliyah**,
  - or **does not include Aliyah**.
- These two cases are **mutually exclusive** and cover all possibilities.

Pascal's Formula: Counting by Cases

- **Case 1: Aliyah is included**
  - We must choose $p-1$ additional people from the remaining $n-1$.
  - Number of possibilities: $\binom{n-1}{p-1}$.
- **Case 2: Aliyah is not included**
  - We choose all p people from the remaining $n-1$.
  - Number of possibilities: $\binom{n-1}{p}$.
- Since the two cases are mutually exclusive and exhaustive, we can apply the **addition principle**:

$$\binom{n}{p} = \underbrace{\binom{n-1}{p-1}}_{\text{case 1}} + \underbrace{\binom{n-1}{p}}_{\text{case 2}}.$$

- This explains how combinations of size p from n are built from smaller problems.

2.8. Newton's Binomial Formula

The Binomial Formula

- For any real numbers a, b and any integer $n \geq 0$,

$$(a+b)^n = \sum_{p=0}^{n} \binom{n}{p} a^p b^{n-p}.$$

Why is it called the binomial formula?

- *A **monomial** is a single term, such as a or b.*
- *The sum of two monomials, $a+b$, is called a **binomial**.*
- *The formula that gives the expansion of $(a+b)^n$ is called the **binomial formula**.*

Distributivity in a Product

Consider the product:

$$(a+b+c+d)(e+f)(g+h+i).$$

- The first factor contains 4 monomials,
- the second contains 2 monomials,
- the third contains 3 monomials.
- In each factor, we choose one monomial and multiply them.

$$(a+b+c+d)(e+f)(g+h+i) = \underbrace{aeg + aeh + aei + afg + \cdots + dfi}_{4\times2\times3=24 \text{ monomials}}.$$

By the multiplication principle, the expanded expression contains $4 \times 2 \times 3 = 24$ monomials.

Expanding $(a+b)^5$

We now apply the same reasoning to

$$(a+b)^5 = (a+b)(a+b)(a+b)(a+b)(a+b).$$

- Each factor contains 2 monomials.
- From each factor, we choose either a or b.
- Each choice produces one monomial in the expansion.

$$\begin{aligned}(a+b)^5 &= \underbrace{aaaaa + aaaab + \cdots + aaabb + \cdots + aabbb + \cdots + abbbb + \cdots + bbbbb}_{2^5=32 \text{ monomials}} \\ &= a^5 + a^4b + \cdots + a^3b^2 + \cdots + a^2b^3 + \cdots + ab^4 + \cdots + b^5.\end{aligned}$$

Grouping Like Terms

$$(a+b)^5 = a^5 + a^4b + \cdots + a^3b^2 + \cdots + a^2b^3 + \cdots + ab^4 + \cdots + b^5$$

- a^5 and b^5 only appear once, but the other monomials appear multiple times, as anagrams:
  - for example, a^4b appears 5 times, as $abbbb$, $babbb$, $bbabb$, $bbbab$, $bbbba$,
  - a^3b^2 appears in the $\binom{5}{2}$ anagrams of $aaabb$.
- Hence:

$$\begin{aligned}(a+b)^5 &= \binom{5}{0}a^5b^0 + \binom{5}{1}a^4b^1 + \binom{5}{2}a^3b^2 + \binom{5}{3}a^3b^3 + \binom{5}{4}a^1b^4 + \binom{5}{5}a^0b^5 \\ &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \\ &= \sum_{p=0}^{5}\binom{5}{p}a^pb^{5-p}.\end{aligned}$$

The Binomial Formula

Proposition 1. Binomial Formula

For any real numbers a, b and any integer $n \geq 0$,

$$(a+b)^n = \sum_{p=0}^{n} \binom{n}{p} a^p b^{n-p}.$$

- The binomial coefficients $\binom{n}{p}$ count how many times each monomial $a^{n-p}b^p$ appears in the expansion, hence the name.

Formal Proof

The general formula can be proved by induction on n, using Pascal's formula.

The Binomial Formula: A Particular Case when $a = b = 1$

Consider the set $[\![1, n]\!] = \{1, 2, \ldots, n\}$.

- The total number of subsets of $[\![1, n]\!]$ is 2^n (see Chapter 1, $\text{card}(\mathcal{P}(S)) = 2^n$ if $\text{card}(S) = n$).
- These subsets can be grouped according to their **size**:
  - 1 empty subset (size 0),
  - n subsets with 1 element,
  - $\binom{n}{2}$ subsets with 2 elements,
  - $\vdots$
  - $\binom{n}{n}$ subsets with n elements.
- Summing over all possible sizes,

$$1 + n + \binom{n}{2} + \cdots + \binom{n}{n} = \sum_{p=0}^{n} \binom{n}{p} = 2^n.$$

2.9. Vandermond's Identity

Vandermonde's Identity

Vandermonde's Identity expresses a binomial coefficient as a sum of products of binomial coefficients.

Proposition 2. Vandermonde's Identity

Let $m, n \in \mathbb{N}^\star$ *and* $0 \leq p \leq n + m$

$$\binom{m+n}{p} = \sum_{k=0}^{p} \binom{m}{k} \binom{n}{p-k}.$$

This identity is the combinatorial basis of the **hypergeometric distribution**, and it explains probabilities of drawing a given number of successes from a finite population.

Vandermonde's Identity: Intuition

- Consider a group of $m+n$ people, divided into two groups:
  - m people in group A,
  - n people in group B.
- We want to choose a committee of size p from the $m+n$ people.
  - **One way** to choose the committee:
    - directly choose p people from $m+n$:

$$\binom{m+n}{p}.$$

  - **Another way**:
    - choose k people from group A (out of m),
    - choose $p-k$ people from group B (out of n).

Vandermonde's Identity: Intuition for the Second Way

For a fixed k, the number of such committees is $\binom{m}{k}\binom{n}{p-k}$.

Why ?

- Fix an integer $k \in \{0, 1, \ldots, p\}$. We want to form a committee of size p such that:
  - exactly k members come from group A (of size m),
    - Number of ways to choose k people from group A: $\binom{m}{k}$.
  - exactly $p - k$ members come from group B (of size n).
    - Number of ways to choose $p - k$ people from group B: $\binom{n}{p-k}$.
  - These two choices are made independently.
- By the multiplication principle, the number of such committees is:

$$\binom{m}{k}\binom{n}{p-k}.$$

2.10. Wrap-Up

Wrap-up: Common Counting Problems

Problem	**Short Description**	**Formula**
Orderings (permutations) of n distinct objects	Use all n objects once; order matters	$n!$
Arrangements	Choose p out of n; no repetition; order matters	$A_n^k = \frac{n!}{(n-p)!}$
Sequences with repetition	Length p; each position has n choices; repetition allowed	n^p
Combinations	Choose p out of n; order does not matter	$\binom{n}{p} = \frac{n!}{p!(n-p)!}$
Permutations of a multiset	n objects with identical groups of sizes $n_1, \ldots, n_k$	$\frac{n!}{n_1! \cdots n_k!}$

- For permutations/arrangements/sequences, **order matters**.
- For combinations, **order does not matter**.

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