Probability

2. Random Experiment, Outcomes, Events

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Foreword

- This short chapter presents a few basic definitions and develops concepts to provide a structure for defining probabilities:
  1. Random experiment,
  2. Outcomes,
  3. Events.
- The content is mainly built following the third chapter from Newbold et al. (2023).

1. Random Experiment and Outcomes

Random Experiment

Definition 1. Random Experiment (Newbold et al., 2023)

*A **random experiment** is a process leading to two or more possible outcomes, without knowing exactly which outcome will occur.*

Examples of random experiments

1. *A coin is tossed and the outcome is either a head or a tail.*
2. *A firm submits 5 bids for public procurement contracts this month. One can observe how many bids that firm wins.*
3. *The number of costumers calling a hotline is not known in advance.*
4. *The daily change in oil price is not known in advance either.*
5. *Selecting a product from a production line and recording whether it meets regulatory standards.*

Outcomes of a Random Experiment

- All possible outcomes of the experiment can be listed.
- Each possible outcome is called a **basic outcome**.

Example of the firm bidding for contracts

- *A firm submits bids for 5 contracts.*
- *It can win $0, 1, 2, 3, 4,$ or 5 contracts.*
- *Each of these six possibilities is a basic outcome.*

Sample Space

Definition 2. Sample Space (Newbold et al., 2023)

The possible outcomes from a random experiment are called the ***basic outcomes****, and the set of all basic outcomes is called the* ***sample space****. We use the symbol Ω to denote the sample space.*

Examples of Random Experiments (1/2)

Examples: Finite sample space

*Consider the random experiment consisting of rolling a fair six-sided die. The **sample space**, that is, the set of all possible outcomes, is*

$$\Omega = \{1, 2, 3, 4, 5, 6\}.$$

*Consider the random experiment consisting of tossing a coin and observing the side on which it lands. The **sample space** is*

$$\Omega = \{\textit{Heads}, \textit{Tails}\}.$$

Examples of Random Experiments (2/2)

Examples: Infinite sample spaces

Consider the random experiment consisting of tossing the coin from the previous slide repeatedly until a head is obtained, and recording the number of tosses required. The ***sample space*** *is the set of positive integers:*

$$\Omega = \{1, 2, 3, \dots\} = \mathbb{N}^*.$$

The sample space may also be continuous. For example, if we observe the waiting time of a person collecting a parcel at the post office, the ***sample space*** *is:*

$$\Omega = (0, +\infty).$$

2. Events

Event

Definition 3. Event (Newbold et al., 2023)

*An **event**, E, is any subset of basic outcomes from the sample space Ω. An event occurs if the random experiment results in one of its constituent basic outcomes. The null event, denoted by $\varnothing$, represents the absence of a basic outcome.*

Example: Firm Bidding For Contracts

Let $\Omega = \{0, 1, 2, 3\}$.

- $E = \{2, 3\}$*: the firm wins at least two contracts.*
- $\varnothing$*: the firm wins a negative number of contracts.*

Intersection of Events

Definition 4. Intersection

*Let A and B be two events. The **intersection** $A \cap B$ is the event that both A and B occur.*

Example

- $A = \{1, 2, 3\}$: *the firm wins at least one contract.*
- $B = \{2, 3\}$: *the firm wins at least two contracts.*
- $A \cap B = \{2, 3\}$: *the firm wins at least two contracts.*

Mutually Exclusive Events

Definition 5. Mutually exclusive events

*Two events A and B are **mutually exclusive** if*

$$A \cap B = \varnothing.$$

Example

- $A = \{0\}$: *the firm wins no contracts.*
- $B = \{3\}$: *the firm wins all contracts.*
- *These events are mutually exclusive.*

Union of Events

Definition 6. Union

*The **union** of two events A and B, denoted $A \cup B$, is the event that at least one of the two events occurs.*

Example

- $A = \{0\}$: *the firm wins no contracts.*
- $B = \{3\}$: *the firm wins all contracts.*
- $A \cup B = \{0, 3\}$: *the firm wins none or all contracts.*

Collectively Exhaustive Events

If the union of several events covers the entire sample space, Ω, we say that these events are collectively exhaustive.

Definition 7. Collectively exhaustive events

A collection of events $\{A_1, \ldots, A_n\}$ is ***collectively exhaustive*** *if*

$$\bigcup_{k=1}^{n} A_k = A_1 \cup \ldots \cup A_n = \Omega.$$

Example

- $A_1 = \{0\}$, $A_2 = \{1\}$, $A_3 = \{2\}$, $A_4 = \{3\}$.
- *Exactly one of these events occurs.*

Complement of an Event

Definition 8. Complement

*The **complement** of an event A, denoted A^c, is the event that A does not occur.*

Example

- $A = \{2, 3\}$: *the firm wins at least two contracts.*
- $A^c = \{0, 1\}$: *the firm wins at most one contract.*

Event and Power Set

- Recall that an **event** is any subset of basic outcomes from the sample space Ω.
- Let Ω be a sample space. We denote by $\mathcal{P}(\Omega)$ the **power set** of Ω, that is, the set of all subsets of Ω (see Chapter 1).
- If Ω is a finite set, we denote by $\text{card}(\Omega)$ the cardinality of Ω, that is, the number of elements in Ω (see also Chapter 1).

Example

Example: Power set and events

Suppose a firm submits bids for three independent contracts. Let $\Omega = \{1, 2, 3\}$, *where the outcome represents the number of contracts won.*
The power set of Ω *is:*

$$\mathcal{P}(\Omega) = \{\{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \varnothing\}.$$

Each element of $\mathcal{P}(\Omega)$ *corresponds to a possible event. For example:*

- $\{1\}$*: the firm wins exactly one contract;*
- $\{2, 3\}$*: the firm wins at least two contracts;*
- $\{1, 2, 3\}$*: the firm wins at least one contract;*
- $\varnothing$*: the impossible event.*

Do we really need all subsets? (1/2)

- The power set assumes we can meaningfully talk about every conceivable event. **In practice, we want to restrict attention to a manageable and coherent collection of events**.
- For example, assume the sample space to be $\Omega = [0, 100]$, which can represent, for example, an individual's annual income in thousands euros.
  - Recall that an event is, by definition, a subset of Ω.
  - The power set treats all subsets as equally admissible, regardless of whether they have any economic interpretation.
  - However, with an infinite space, some of the subsets are:
    - **intuitive**: e.g., intervals ($A = [0, 30]$, $B = (50, 80]$, $\ldots$), finite unions of intervals, or complements of some subsets,
    - **unintuitive**: e.g., ultra-fine distinctions ($E = \{42.137491928\}$), incomes (measured in euros) whose last decimal digit is even ($E = \{x : \text{the last digit of } x \text{ is even}\}$), ...
  - So while all subsets exist mathematically, they are not all economically meaningful events for the purposes of a given model.

Do we really need all subsets? (2/2)

- In the income example with $\Omega = [0, 100]$, suppose we decide that the only economically relevant question is which income bracket applies:
  - below 30 (low income),
  - between 30 and 70 (middle income),
  - above 70 (high income).
- This defines three basic events:
  - $A_1 = [0, 30), \quad A_2 = [30, 70), \quad A_3 = [70, 100].$
- What are all the events we decide to keep track of?
  - each bracket individually,
  - any union of brackets,
  - complements of those unions,
  - the empty set and the whole space.
- This finite collection is all the events we allow ourselves to reason about.
- All other subsets of Ω, while mathematically legitimate, play no role in our analysis.

A coherent collection of events

- In practice, as illustrated in the previous example, we do not want to consider all subsets of Ω.
- Instead, we choose a collection of events that:
  - includes the basic distinctions we care about,
  - allows us to take negations ("not A"),
  - allows us to combine events ("A or B").
- A collection of subsets of Ω with these properties is called a **σ-algebra**.

Definition 9. σ-algebra

Let Ω be a sample space. A collection $\mathcal{F}$ of subsets of Ω is called a ***σ-algebra*** *if:*

- $\Omega \in \mathcal{F}$;
- *if* $A \in \mathcal{F}$, *then* $A^c \in \mathcal{F}$;
- *if* $(A_n)_{n \geq 1}$ *is a sequence of sets in* $\mathcal{F}$, *then* $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

The sets in $\mathcal{F}$ are called the events we can reason about.

The σ-algebra generated by income brackets

- Consider again the income space $\Omega = [0, 100]$ and the three basic events:
  - $A_1 = [0, 30)$ (low income),
  - $A_2 = [30, 70)$ (middle income),
  - $A_3 = [70, 100]$ (high income).
- Suppose these are the only basic distinctions we care about.
- What is the σ-algebra generated by $\{A_1, A_2, A_3\}$?
- The collection of subsets of Ω, $\mathcal{F}$, generated by $\{A_1, A_2, A_3\}$, consists of all possible unions of these sets:
  - $\varnothing$,
  - $A_1,\ A_2,\ A_3$,
  - $A_1 \cup A_2,\ A_1 \cup A_3,\ A_2 \cup A_3$,
  - $\Omega = A_1 \cup A_2 \cup A_3$.
- These are exactly the events we allow ourselves to reason about.

Checking the σ-algebra axioms for the income brackets example

- Let $\mathcal{F}$ be the collection of events:

$$\mathcal{F} = \{\varnothing,\ A_1,\ A_2,\ A_3,\ A_1 \cup A_2,\ A_1 \cup A_3,\ A_2 \cup A_3,\ \Omega\}.$$

- We now verify that $\mathcal{F}$ is a σ-algebra.
  - **Whole space:** $\Omega \in \mathcal{F}$.
  - **Complements:**
    - $A_1^c = A_2 \cup A_3 \in \mathcal{F}$,
    - $A_2^c = A_1 \cup A_3 \in \mathcal{F}$,
    - $A_3^c = A_1 \cup A_2 \in \mathcal{F}$.
  - **Unions:** the union of any sets in $\mathcal{F}$ is still a union of brackets and hence belongs to $\mathcal{F}$.
- Therefore, $\mathcal{F}$ is a σ-algebra on Ω.

Notes:

- A σ-algebra can be interpreted as the information available to an observer.
- Events in $\mathcal{F}$ are those that can be distinguished given this information.

Remarks

- When Ω is finite, we will always use the full σ-algebra, i.e., $\mathcal{P}(\Omega)$.
- In the remainder of this course, we will assume that the underlying sample space is equipped with a σ-algebra $\mathcal{F}$, which will not be specified explicitly.

An exemple with a finite sample space

Let $\Omega = \{1, 2, 3\}$.

- *The power set $\mathcal{P}(\Omega)$ is a σ-algebra.*
- *$\{\varnothing, \Omega\}$ is the smallest σ-algebra.*
- *$\{\varnothing, \{1, 2\}, \{3\}, \Omega\}$ is also a σ-algebra.*

Exercise

Exercise (adapted from Newbold et al., 2023)

A shipwright on the Grand Line has a small dock with five ships: three Marine ships (M_1, M_2, M_3) and two Pirate ships (P_1, P_2). Two pirates, Luffy and Nami, arrive independently at the dock. Each of them secretly selects one ship. They do not coordinate or communicate.

Let the events A and B be defined as follows:

- A: *At least one Pirate ship is selected.*
- B: *Both characters select ships of the same type.*

1. *List all ordered pairs of ships in the sample space (assume Luffy choses first).*
2. *Define event A explicitly as a subset of the sample space.*
3. *Define event B explicitly as a subset of the sample space.*
4. *Define the complement of A.*
5. *Show that $(A \cap B) \cup (A^c \cap B) = B$.*
6. *Show that $A \cup (A^c \cap B) = A \cup B$.*

Solution (1/6)

- All the ships available are: M_1, M_2, M_3, P_1, P_2.
- For convenience, let us assume that Luffy chooses first and Nami choses second.
- Each outcomes is an ordered pair: $(\text{Luffy's choice}, \text{Zoro's choice})$.

1. *List all ordered pairs of ships in the sample space*.
   The sample space is the set of all ordered pairs of ships:

$$\Omega = \{(x, y) : x, y \in \{M_1, M_2, M_3, P_1, P_2\}\},$$

which gives $5 \times 5 = 25$ outcomes:

$$\begin{aligned}\Omega = \{&(M_1, M_1), (M_1, M_2), (M_1, M_3), (M_1, P_1), (M_1, P_2),\\ &(M_2, M_1), (M_2, M_2), (M_2, M_3), (M_2, P_1), (M_2, P_2),\\ &(M_3, M_1), (M_3, M_2), (M_3, M_3), (M_3, P_1), (M_3, P_2),\\ &(P_1, M_1), (P_1, M_2), (P_1, M_3), (P_1, P_1), (P_1, P_2),\\ &(P_2, M_1), (P_2, M_2), (P_2, M_3), (P_2, P_1), (P_2, P_2)\}.\end{aligned}$$

Solution (2/6)

2. *Define event A explicitly as a subset of the sample space.*
   (A: At least one Pirate ship is selected)
   Thus, A consists of all outcomes where at least one of the two ships is either P_1 or P_2:

$$\begin{aligned} A = \{&(M_1, P_1), (M_1, P_2), \\ &(M_2, P_1), (M_2, P_2), \\ &(M_3, P_1), (M_3, P_2), \\ &(P_1, M_1), (P_1, M_2), (P_1, M_3), (P_1, P_1), (P_1, P_2), \\ &(P_2, M_1), (P_2, M_2), (P_2, M_3), (P_2, P_1), (P_2, P_2)\}. \end{aligned}$$

   More formally:

$$A = \{(x, y) \in \Omega : x \in \{P_1, P_2\} \text{ or } y \in \{P_1, P_2\}\}.$$

Solution (3/6)

3. *Define event B explicitly as a subset of the sample space.*
   (B: Both characters select ships of the same type (both Pirate, or both Marine ships))

$$
\begin{aligned}
B = \{&(M_1, M_1), (M_1, M_2), (M_1, M_3),\\
&(M_2, M_1), (M_2, M_2), (M_2, M_3),\\
&(M_3, M_1), (M_3, M_2), (M_3, M_3)\\
&(P_1, P_1), (P_1, P_2), (P_2, P_1), (P_2, P_2)\}
\end{aligned}
$$

There are $9 + 4 = 13$ outcomes in B. More formally:

$$
\begin{aligned}
B = &\{(M_i, M_j) : i, j \in \{1, 2, 3\}\}\\
&\cup \{(P_i, P_j) : i, j \in \{1, 2\}\}.
\end{aligned}
$$

Solution (4/6)

4. *Define the complement of A.*
(A: At least one Pirate ship is selected)

$$A^c = \{\text{no Pirate ship is selected}\}$$

This means that both ships must be Marine ships. Hence,

$$\begin{aligned}A^c = \{&(M_1, M_1), (M_1, M_2), (M_1, M_3),\\ &(M_2, M_1), (M_2, M_2), (M_2, M_3),\\ &(M_3, M_1), (M_3, M_2), (M_3, M_3)\}.\end{aligned}$$

There are 9 outcomes in A^c.

Note: you can observe that $A \cup A^c = \Omega$ and that $A \cap A^c = \varnothing$.

Solution (5/6)

5. *Show that* $(A \cap B) \cup (A^c \cap B) = B$.
   (A: At least one Pirate ship is selected)
   (B: Both characters select ships of the same type)
   - $A \cap B$: both ships are Pirate ships.

$$A \cap B = \{(P_1, P_1), (P_1, P_2), (P_2, P_1), (P_2, P_2)\}.$$

   - $A^c \cap B$: both ships are Marine ships.

$$\begin{aligned} A^c \cap B = A^c = \{&(M_1, M_1), (M_1, M_2), (M_1, M_3), \\ &(M_2, M_1), (M_2, M_2), (M_2, M_3), \\ &(M_3, M_1), (M_3, M_2), (M_3, M_3)\}. \end{aligned}$$

   - Taking the union:

$$(A \cap B) \cup (A^c \cap B) = B.$$

Solution (6/6)

6 *Show that* $A \cup (A^c \cap B) = A \cup B$.
(A: At least one Pirate ship is selected)
(B: Both characters select ships of the same type)

- Recall from Chapter 1 that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
- Hence:

$$\begin{aligned} A \cup (A^c \cap B) &= (A \cup A^c) \cap (A \cup B) \\ &= \Omega \cap (A \cup B) \\ &= A \cup B. \end{aligned}$$

References I

Newbold, P., Carlson, W. L. and Thorne, B. M. (2023). *Statistics for Business and Economics, 10th Global Edition*. Pearson Education Limited.