Probability

1. Reminders About Sets and Spaces

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Foreword

- This first chapter is not about probabilities.
- It provides some reminders for fundamental notions and notations used in probabilities:
  - sets and sample spaces,
  - cardinality,
  - unions and intersections.
- The content follows the first chapter from Elsner Bernhard (2021).

1. Sets and Cardinality

Sets and subsets

From your high-school years, recall that:

- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of natural numbers,
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of integers,
- $\mathbb{Q}$ is the set of rational numbers, such as $\frac{1}{2}$, $\frac{4}{1}$, $\frac{-2}{9}$
- $\mathbb{R}$ is the set of real numbers (which contains numbers such as $\sqrt{2}$, π, ...).

Definition of a Set

Definition 1. Set

- *A* ***set*** *is a* ***collection of objects****, called* ***elements****.*
- *The elements of a set do* ***not*** *have to be numbers.*
- *A set is* ***unordered****: the order of elements does not matter.*

Example

$$S = \{a, b, c\}$$

is the set containing the three elements a, b, and c.

- The membership of an element a in a set S is denoted by:

$$a \in S.$$

Subsets

Let A and B be two sets.

- If every element of A is also an element of B:
  - we write $A \subset B$ or $B \supset A$.
  - it reads as A is included in B, or A is a **subset** of B, or B is a **superset** of A.
- Expressed in notations:

$$A \subset B \quad \text{iff} \quad \forall x (x \in A \Rightarrow x \in B).$$

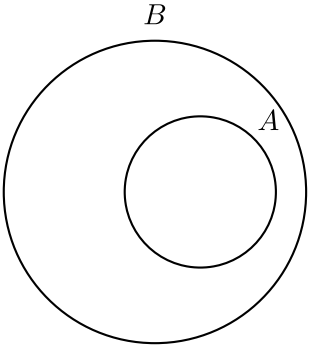
- $A \subset B$ does **not** imply that $A \neq B$.
- If $A \subset B$ and $B \subset A$, then $A = B$.

Examples

- *Finite sets:* $\{a, b, c\} \subset \{a, b, c, d\}$
- *Standard numerical sets:*

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

- The inclusion relation $A \subset B$ can be visualized using a Venn diagram.



$A \subset B$

Subsets Defined by a Property

- A subset A of a set B can be defined by a property.
- Notation:

$$A = \{x \in B \mid \mathcal{P}(x)\}$$

- This is read as:
  "A is the set of all elements x in B such that the property $\mathcal{P}(x)$ holds."

Examples

- $\mathbb{R}^* = \{x \in \mathbb{R} \mid x \neq 0\}$ *is the set of non-zero real numbers.*
- $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$ *is the set of non-negative real numbers.*
- $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ *is a bounded closed interval.*
- $[\![a, b]\!] = \{k \in \mathbb{Z} \mid a \leq k \leq b\}$ *is a set of consecutive integers.*

$$[\![2, 5]\!] = \{2, 3, 4, 5\}.$$

Cardinality

Definition 2. Cardinality

- *The **cardinality** of a set is the number of elements it contains.*
- *It is denoted by card(A) (or $|A|$) for a set A.*
- *If A has a finite number of elements, then card$(A) \in \mathbb{N}$.*

Examples

- *If $A = \{a, b, c\}$, then card$(A) = 3$.*
- *If $B = \{1, 2, 3, 4, 5\}$, then card$(B) = 5$.*
- *The empty set $\varnothing$ has cardinality card$(\varnothing) = 0$.*

Infinite Sets

Definition 3. Infinite Sets

*A set is **infinite** if it does not have a finite number of elements.*

Examples

- $\mathbb{N}$, $\mathbb{Z}$
- $\mathbb{R}$, $[a, b]$ *with* $a < b$

Countable and Uncountable Sets

- Some infinite sets can be **counted**.

Definition 4. Countable Set

*A set is **countable** if it can be put in one-to-one correspondence with $\mathbb{N}$.*

- $\mathbb{N}$ and $\mathbb{Z}$ are countable.
- $\mathbb{R}$ and any interval $[a, b]$ with $a < b$ are uncountable.
- In particular:

$$\text{card}(\mathbb{N}) < \text{card}(\mathbb{R}).$$

Inclusion and Cardinality

- If $A \subset B$ and A is finite, then:

$$\text{card}(A) \leq \text{card}(B).$$

Example

$$A = \{a, b\}, \quad B = \{a, b, c, d\}.$$

Then $A \subset B$, card$(A) = 2$, and card$(B) = 4$.

Inclusion and Cardinality

- Strict inclusion often implies a strictly larger cardinality, but this is not always true for infinite sets.

Example

Consider the sets:

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}, \qquad 2\mathbb{N} = \{0, 2, 4, 6, \ldots\}.$$

- *We have a strict inclusion:* $2\mathbb{N} \subset \mathbb{N}$*.*
- *Nevertheless, both sets have the same cardinality.*
- *Indeed, the mapping* $n \mapsto 2n$ *defines a one-to-one correspondence between* $\mathbb{N}$ *and* $2\mathbb{N}$*.*

A Set Can Contain Sets

- An element of a set does not have to be a number.
- It can also be another set.
- Analogy: folders and files on a computer. A folder can contain:
  - files,
  - other folders
- A folder within the folder counts as **one element**,
- even though it can contain many files.
- Similarly, a set may contain another set as a single element.

Example

Example

Consider the set:

$$S = \{1, \{2, 3\}\}.$$

- The set S has exactly two elements:
  - the number 1,
  - the set $\{2, 3\}$.
- Therefore:

$$\text{card}(S) = 2.$$

- Note that $2 \notin S$ and $3 \notin S$.

Power Set

Definition 5. Power Set

*The **power set** of a set S is the set of all subsets of S. It is denoted by:*

$$\mathcal{P}(S).$$

- By definition, every element of $\mathcal{P}(S)$ is itself a set.
- The empty set $\varnothing$ is a subset of every set.

Examples

- *If $S = \varnothing$, then:*

$$\mathcal{P}(\varnothing) = \{\varnothing\}, \qquad \mathit{card}(\mathcal{P}(\varnothing)) = 1 = 2^0.$$

- *If $S = \{1\}$, then:*

$$\mathcal{P}(\{1\}) = \{\varnothing, \{1\}\}, \qquad \mathit{card}(\mathcal{P}(\{1\})) = 2 = 2^1.$$

Power Set: another example

Example

Consider $S = \{1, 2\}$.

Then:

$$\mathcal{P}(\{1, 2\}) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}.$$

- There are 4 subsets.
- Hence:

$$\text{card}(\mathcal{P}(\{1, 2\})) = 4 = 2^2.$$

Cardinality of the Power Set

Proposition 1. Cardinality of the Power Set

Let S be a finite set. If card$(S) = n$, then:

$$\textit{card}(\mathcal{P}(S)) = 2^n.$$

Sketch of the proof of why card$(\mathcal{P}(S)) = 2^n$

- Each element of S can either: belong to a subset, or not belong to it.
- For each of the n elements, there are 2 choices (binary choices).
- Therefore, the total number of subsets is:

$$2 \times 2 \times \cdots \times 2 = 2^n.$$

Sketch of the proof of why card$(\mathcal{P}(S)) = 2^n$

Example: Let $S = \{1, 2, 3\}$.
To construct a subset of S, we must decide for each element whether it belongs to the subset or not, by answering the following three questions:

1. Does the subset contain the element 1?
2. Does the subset contain the element 2?
3. Does the subset contain the element 3?

For example, for the subset $\{1, 3\}$: yes-no-yes. The number of possible triple-answers is $2 \times 2 \times 2 = 2^3 = 6$.

Subset	1	2	3
$\varnothing$	no	no	no
$\{1\}$	yes	no	no
$\{2\}$	no	yes	no
$\{3\}$	no	no	yes
$\{1, 2\}$	yes	yes	no
$\{1, 3\}$	yes	no	yes
$\{2, 3\}$	no	yes	yes
$\{1, 2, 3\}$	yes	yes	yes

Exercises (some from Elsner Bernhard, 2021)

Exercises

1. *What is the cardinality of* $\{\{\{\bullet\}, \{\star, \square\}\}, \triangle\}$*?*
2. *What is the cardinality of* $[\![-1, 3]\!]$*?*
3. *Let* n *and* m *be two integers such that* $n \leq m$*. What is the cardinality of* $[\![n, m]\!]$*?*
4. *Let* $A = \{1, 3\}, \quad B = \{1, 2, 3, 4\}, \quad C = \{1, 3, 5\} \quad D = \{\{1, 3\}, 4, 5\}$*. Which of the following inclusions are true?*
    1. $A \subset B$,
    2. $A \subset C$,
    3. $B \subset C$,
    4. $A \subset A$,
    5. $C \subset D$.

Solutions

1. $\text{card}\left(\{\{\{\bullet\}, \{\star, \square\}\}, \triangle\}\right) = 2.$
2. $\text{card}\left(\llbracket -1, 3 \rrbracket\right) = \text{card}\left(\{-1, 0, 1, 2, 3\}\right) = 5.$
3. $\text{card}\left(\llbracket n, m \rrbracket\right) = m - n + 1.$
4. 1, 2, 4.

2. Operations on Sets

Operations on Sets

Let A and B be subsets of a set S.

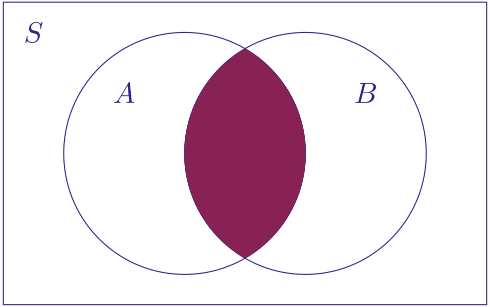
- We define the following operations:
  - intersection,
  - union,
  - difference,
  - complement.
- These operations allow us to combine and manipulate sets.

Intersection

Definition 6. Intersection

*The **intersection** of A and B is the set:*

$$A \cap B = \{x \in S \mid x \in A \text{ and } x \in B\}.$$

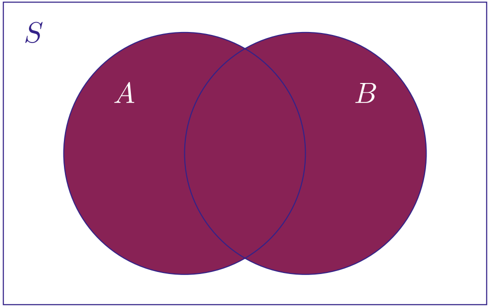


Union

Definition 7. Union

*The **union** of A and B is the set:*

$$A \cup B = \{x \in S \mid x \in A \text{ or } x \in B\}.$$

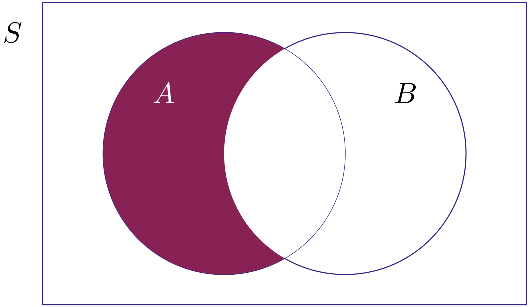


Difference

Definition 8. Difference

*The **difference** of A and B is the set:*

$$\boxed{A \setminus B} = \{x \in S \mid x \in A \text{ and } x \notin B\}.$$

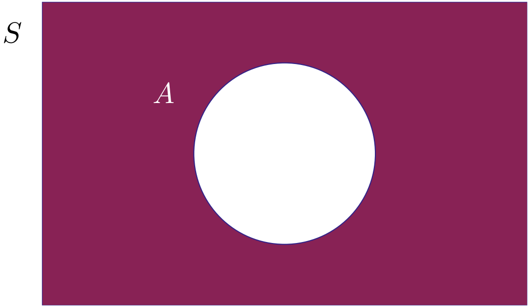


Complement

Definition 9. T

*he **complement** of A (in S) is the set:*

$$A^c = \{x \in S \mid x \notin A\}.$$



Warning: Parentheses Matter!

Consider the expression: $A \cap B \cup C$.

- This expression is **ambiguous**.
- It may mean:
  - $(A \cap B) \cup C$,
  - or $A \cap (B \cup C)$.
- These two sets are generally **not equal**.
- Parentheses are thus essential to avoid ambiguity.

Commutativity

Definition 10. Commutativity of the Union and the Intersection

*The operations **union** and **intersection** are **commutative**.*

- *For any sets A and B:*

$$A \cup B = B \cup A, \qquad A \cap B = B \cap A.$$

- *The order of the sets does not matter.*

Associativity

Definition 11. Associativity of the Union and the Intersection

*The operations **union** and **intersection** are **associative**.*

- *For any sets A, B, and C:*

$$(A \cup B) \cup C = A \cup (B \cup C),$$

$$(A \cap B) \cap C = A \cap (B \cap C).$$

- *Parentheses are not needed when only one operation is used.*

Distributivity

Definition 12. Distributivity of the Union and the Intersection

Union *and* ***intersection*** ***distribute*** *over each other.*

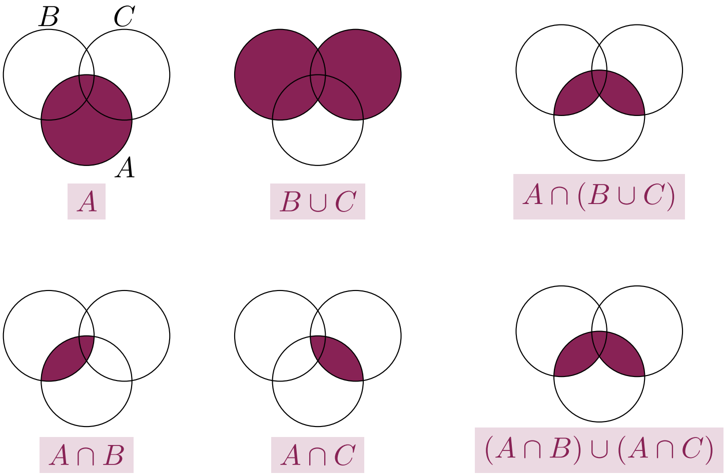
- *For any sets A, B, and C:*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

- *These rules are analogous to distributivity in algebra.*

Distributivity (Venn-diagram proof for $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$)



Dichotomy

Definition 13. Dichotomy

Let A be a subset of a set S.

- *Every element of S is either in A or in its complement A^c.*
- *Formally:*

$$A \cup A^c = S, \qquad A \cap A^c = \varnothing.$$

- *A set and its complement form a partition of S.*

De Morgan's Laws

De Morgan's laws describe how complements interact with unions and intersections.

Proposition 2. De Morgan's Law

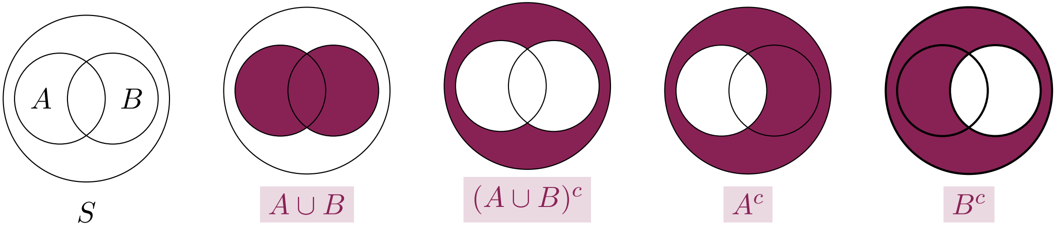
For any sets A and B:

$$(A \cup B)^c = A^c \cap B^c,$$

$$(A \cap B)^c = A^c \cup B^c.$$

Augustus De Morgan (1806–1871), a British mathematician and logician. Source: Sophia Elizabeth De Morgan - Memoir of Augustus De Morgan

De Morgan's Laws: Proof of $(A \cup B)^c = A^c \cap B^c$



Summary of Set Operation Rules

Rule	**Formulation**
Commutativity	$A \cup B = B \cup A \qquad A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$
	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Dichotomy	$A \cup A^c = S \qquad A \cap A^c = \varnothing$
De Morgan's laws	$(A \cup B)^c = A^c \cap B^c$
	$(A \cap B)^c = A^c \cup B^c$

Exercises (some from Elsner Bernhard, 2021)

Exercises

1. *Let* $A = \{-3, \frac{1}{4}, 2, 7, 10, -\frac{5}{2}\}$ *and* $B = \{4, -3, 7, \frac{1}{4}, 9\}$.
   Determine $A \cup B$, $A \cap B$, *and* $A \setminus B$
2. *Let* $A = \{\bullet\}, \quad B = \{\star\}, \quad C = \{\star, \bullet\}$.
   Show that $(A \cap B) \cup C \neq A \cap (B \cup C)$.

Solutions (1/2)

Exercise 1.

- The union contains all distinct elements that are in A or in B:

$$A \cup B = \{-3, \frac{1}{4}, 2, 7, 10, -\frac{5}{2}, 4, 9\}.$$

- The intersection contains elements that are common to both sets:

$$A \cap B = \{-3, \frac{1}{4}, 7\}.$$

- The set difference contains elements that are in A but not in B:

$$2, 10, -\frac{5}{2}.$$

Solutions (2/2)

Exercise 2.

- $(A \cap B) \cup C = \varnothing \cup C = C = \{\star, \bullet\}$
- $A \cap (B \cup C) = A \cap C = \{\bullet\} = A$
- Hence, $(A \cap B) \cup C \neq A \cap (B \cup C)$.

3. Partitions

Disjoint Sets

Definition 14. Disjoin Sets

Let A and B be subsets of a set S.

- *If*
$$A \cap B = \varnothing,$$
*then A and B are said to be **disjoint**.*
- *Disjoint sets have no element in common.*

- In probability, disjoint events are also called **incompatible**.

Definition of a Partition

Definition 15. Partition

*A **partition** (or **complete system**) of a set S is a collection of subsets $\{A_1, A_2, \ldots, A_n\}$ such that:*

- *$A_k \neq \varnothing$ for all k (Non-empty),*
- *$A_k \cap A_j = \varnothing$ whenever $k \neq j$ (pairwise disjoint),*
- *$S = A_1 \cup A_2 \cup \cdots \cup A_n$ (cover).*

Interpretation: If $\{A_1, A_2, \ldots, A_n\}$ is a partition of S,

- every element of S belongs to **exactly one** of the sets A_k,
- the sets do not overlap,
- together, they exhaust the whole set S.

Examples of Partitions

- The 16 administrative districts of Marseille form a partition of the city of Marseille.

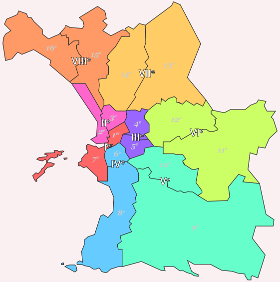


Figure 1: Arrondissements in Marseille. Source: Superbenjamin, Wikipedia.

- The collection $\{\{0, 2\}, \{1\}, \{3, 4\}\}$ is a partition of $[\![0, 4]\!] = \{0, 1, 2, 3, 4\}$.

Partition Induced by a Subset

Let $A \subset S$ with $A \neq \varnothing$ and $A \neq S$.

- Then:

$$\{A, A^c\}$$

  is a partition of S.

Trivial Partitions

- The collection $\{S\}$ is a partition of S.
  - this is a trivial partition.
- The collection $\{\{x\} \mid x \in S\}$ is also a partition of S.
  - each subset contains exactly one element.

Why Partitions Matter in Probabilities?

- Partitions allow us to decompose a set into mutually exclusive cases.
- In probability, they will be essential for:
  - the law of total probability,
  - conditioning,
  - Bayesian reasoning.

Exercises

Exercises

1. *Let $S = \{1, 2, 3, 4, 5, 6\}$. For each collection below, decide whether it is a partition of S. Justify by checking the three conditions (non-empty, disjoint, union in S).*
   1. $\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$
   2. $\{\{1, 2, 3\}, \{3, 4\}, \{5, 6\}\}$
   3. $\{\{1, 2\}, \{3, 4\}\}$
   4. $\{\{1, 2\}, \varnothing, \{3, 4, 5, 6\}\}$
2. *Find all partitions of the set $\{\circ, \triangle, \square\}$.*

Solutions (1/3)

Exercise 1. $S = \{1, 2, 3, 4, 5, 6\}$. A collection $\{A_1, \ldots, A_n\}$ is a partition of S iff:

1. Non-empty: $A_k \neq \varnothing$ for all k,
2. Pairwise disjoint: $A_i \cap A_j = \varnothing$ for $i \neq j$,
3. Cover: $A_1 \cup \ldots \cup A_n = S$.

Hence:

1. $\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$: this is a partition of S:
   - Non-empty: yes, $A_k \neq \varnothing$ for all k
   - Pairwise disjoint: yes, $A_i \cap A_j = \varnothing$ for $i \neq j$,
   - Cover: $\{1, 2\} \cup \{3, 4\} \cup \{5, 6\} = \{1, 2, 3, 4, 5, 6\} = S$

Solutions (2/3)

Exercise 1 (contd). $S = \{1, 2, 3, 4, 5, 6\}$.

2. $\{\{1, 2, 3\}, \{3, 4\}, \{5, 6\}\}$: this is **not** a partition of S:
   - Non-empty: yes, $A_k \neq \varnothing$ for all k
   - Pairwise disjoint: $\{1, 2, 3\} \cap \{3, 4\} = \{3\} \neq \varnothing$.
3. $\{\{1, 2\}, \{3, 4\}\}$: yes, this is **not** a partition of S:
   - Non-empty: yes, $A_k \neq \varnothing$ for all k
   - Pairwise disjoint: yes, $A_i \cap A_j = \varnothing$ for $i \neq j$,
   - Cover: $\{1, 2\} \cup \{3, 4\} = \{1, 2, 3, 4\} \neq S$
4. $\{\{1, 2\}, \varnothing, \{3, 4, 5, 6\}\}$: this is **not** a partition of S:
   - Non-empty: no, $\varnothing$ is included.

Solutions (3/3)

Exercise 2. $\{\circ, \triangle, \square\}$

- Trivial partitions:
  - $\{\{\circ, \triangle, \square\}\}$
  - $\{\{\square\}, \{\triangle\}, \{\circ\}\}$
- Non-trivial partitions:
  - $\{\{\circ\}, \{\triangle, \square\}\}$
  - $\{\{\triangle\}, \{\circ, \square\}\}$
  - $\{\{\square\}, \{\triangle, \circ\}\}$

3.1. Partitions and Cardinality

Disclaimer

Warning

We will use $|A|$ instead of card(A) in this part of the course to make mathematical notations visually easier to understand.

Cardinality of a Union

Definition 16. Cardinality of a Union (cardinality version of the inclusion–exclusion identity)

*Let A and B be **finite** subsets of a set S.*

- *The cardinality of the union satisfies:*

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

- *The term $|A \cap B|$ avoids counting common elements twice.*

Tip

This also means that

$$|A| + |B| = |A \cup B| + |A \cap B|.$$

Union of Disjoint Sets

Proposition 3. Union of Disjoint Sets

Let A and B be finite subsets of S.

- *If $A \cap B = \varnothing$, then:*
$$|A \cup B| = |A| + |B|.$$

Proof.

- This is a direct consequence of the general formula.
- If $A \cap B = \varnothing$, then $|A \cap B| = 0$.

Example

Example

Let:

$$A = \{1, 2, 3, 4\}, \qquad B = \{3, 4, 5, 6\}.$$

- $|A| = 4, \quad |B| = 4$
- $A \cap B = \{3, 4\}$, so $|A \cap B| = 2$
- Therefore:

$$|A \cup B| = 4 + 4 - 2 = 6.$$

Exercise

Exercise

A labour market survey is conducted among 1,000 individuals. Let:

- A *be the set of individuals who are* ***employed****,*
- B *be the set of individuals who have a* ***university degree****.*

The survey reports:

- $|A| = 650$ *individuals are employed,*
- $|B| = 520$ *individuals have a university degree,*
- $|A \cup B| = 820$ *individuals are either employed or have a university degree (or both).*

1. *Compute the number of individuals who are both employed and have a university degree.*
2. *How many individuals are neither employed nor have a university degree?*

Solution (1/2)

Let S denote the set of the $1{,}000$ individuals surveyed. A and B are two finite subsets of S. We use the identity:

$$|A \cap B| = |A| + |B| - |A \cup B|.$$

1. Number of individuals who are both employed and have a university degree:

$$|A \cap B| = 650 + 520 - 820 = 350.$$

2. Number of individuals who are neither employed nor have a university degree:

$$|(A \cup B)^c| = |S| - |A \cup B| = 1000 - 820 = 180.$$

Individuals who are both employed and educated are counted twice in $|A| + |B|$, which explains why we subtract $|A \cap B|$.

Solution: Using a Table (2/2)

The exercise instructions state:

	B	B^c	Total
A	?	?	650
A^c	?	?	?
Total	520	?	1000

We can complete:

	B	B^c	Total
A	350	300	650
A^c	170	180	350
Total	520	480	1000

Cardinality and Partitions

Proposition 4. Cardinality and Partitions

Let $\{A_1, A_2, \ldots, A_n\}$ be a partition of a ***finite*** *set S.*

- *The sets $A_1, \ldots, A_n$ are pairwise disjoint.*
- *Their union is equal to S.*
- *Therefore:*

$$|S| = \sum_{i=1}^{n} |A_i|.$$

- A partition splits S into non-overlapping pieces.
- Each element of S is counted exactly once.
- The total cardinality is the sum of the cardinalities of the parts.

Cardinality of a Union (General Argument)

Let A and B be subsets of a set S (not necessarily finite).

- The sets $A \setminus B$ and B are disjoint.
- In addition:
$$(A \setminus B) \cup B = A \cup B.$$
- Hence, $\{A \setminus B, B\}$ is a partition of $A \cup B$.

Therefore,

$$|A \cup B| = |A \setminus B| + |B|.$$

A Second Partition

Observe that:

$$A = (A \setminus B) \,\dot{\cup}\, (A \cap B).$$

(where $\dot{\cup}$ denotes the disjoint union)

- The sets $A \setminus B$ and $A \cap B$ are disjoint.
- Their union is equal to A.
- Hence, $\{A \setminus B,\, A \cap B\}$ is a partition of A.

Therefore,

$$|A| = |A \setminus B| + |A \cap B|.$$

Combining the Two Equalities

From the previous slides, we have:

$$|A \cup B| = |A \setminus B| + |B|,$$

$$|A| = |A \setminus B| + |A \cap B|.$$

Adding $|A \cap B|$ to the first equality gives:

$$|A \cap B| + |A \cup B| = |A| + |B|.$$

Rearranging the terms:

$$|A| + |B| = |A \cup B| + |A \cap B|.$$

Finite Case (Inclusion–Exclusion Formula)

- If A and B are **finite** sets, we may subtract $|A \cap B|$.
- We obtain the familiar formula:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

References I

Elsner Bernhard, B. (2021). *Voyage au pays des probas: Cours et exercices corrigés*. Édition Marketing Ellipses.